Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Absolute stability analysis of Lure's systems and some extensions

Guang Li
School of Engineering, Computing and Mathematics, Exeter

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Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Outline

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
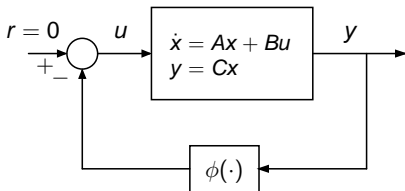Bibliography

Lure's systems

Figure: Lure System

Definition – Lure system

A linear time invariant (LTI) plant is connected with a nonlinearity:

- The plant $G(s) := (A, B, C, 0)$ is a minimal realization.
- The nonlinearity $\phi(\cdot)$ is memoryless, locally Lipschitz in y and satisfies a sector (bound) condition.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Sector bound condition – SISO case

Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ with $\phi(0) = 0$. We denote $\phi \in [k_1, k_2]$ if $\phi(\cdot)$ satisfies

$$[\phi(y) - k_1 y][\phi(y) - k_2 y] \leq 0 \quad \forall y \in \mathbb{R} \tag{1}$$

Figure: Sector bound condition (SISO case)

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Sector bound condition – SISO case

Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ with $\phi(0) = 0$. We denote $\phi \in [k_1, k_2]$ if $\phi(\cdot)$ satisfies

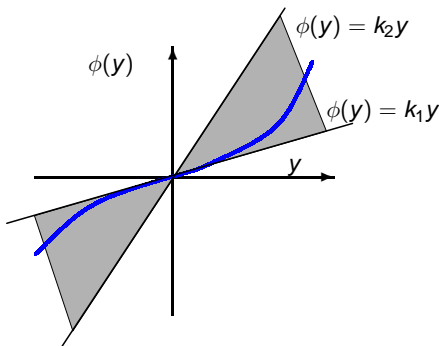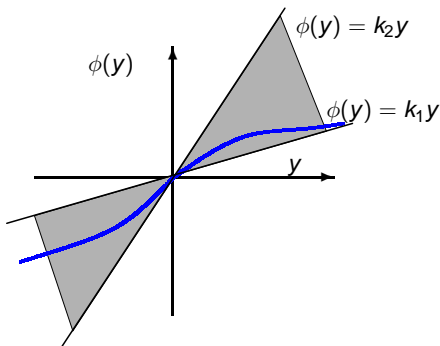$$[\phi(y) - k_1 y][\phi(y) - k_2 y] \leq 0 \quad \forall y \in \mathbb{R} \quad (1)$$

Figure: Sector bound condition (SISO case)

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
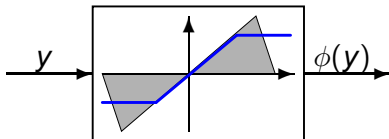Bibliography

Common nonlinearites

Saturation

Figure: Saturation

Saturation nonlinearity satisfies:

$$\phi(y)(\phi(y) - y) \leq 0$$

which corresponds to the general case when $k_1 = 0$ and $k_2 = 1$. Denote $\phi \in [0, 1]$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
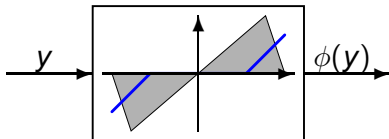Conclusion
Bibliography

Common nonlinearites

Deadzone

Figure: Deadzone

Deadzone nonlinearity satisfies:

$$\phi(y)(\phi(y) - y) \le 0$$

which corresponds to the general case when $k_1 = 0$ and $k_2 = 1$. Denote $\phi \in [0, 1]$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Sector bound condition – Multivariable Case

- Diagonal case:

$$[\phi(y) - K_1 y]^T [\phi(y) - K_2 y] \leq 0$$

  where K_1 and K_2 are diagonal matrices with $K_1 - K_2 < 0$ and

$$
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}
\qquad
\phi(y) = \begin{bmatrix} \phi_1(y_1) \\ \phi_2(y_2) \\ \vdots \\ \phi_m(y_m) \end{bmatrix}
$$

- Coupled case:

$$[\phi(y) - K_1 y]^T [\phi(y) - K_2 y] \leq 0$$

  where K_1 and K_2 are positive definite matrices with $K_1 - K_2 < 0$, and $\phi_i(y)$ can depend on y_j.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
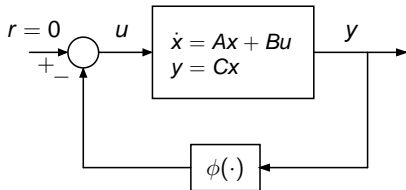Bibliography

A Lure's system

Figure: A Lure System

Definition – Absolute stability [Khalil(2000)]

Suppose ϕ satisfies a sector condition. The system is absolutely stable if the equilibrium point at the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Outline

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Some Classical Stability Criteria

There are a number of classical criteria for the absolute stability analysis of the Lure system:

- Circle criterion;
- Popov criterion;
- Zames-Falb multiplier theory [Zames and Falb(1968)];
- Small gain;
- Passivity theory;
- · · ·

These criteria are related to each other.
Two key problems to investigate:

- Less conservative?
- Easier to use?

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

the S-procedure

the S-procedure [Yakubovich(1971)]

Suppose there are two statements as:

$$S1: \sigma_1 \geq 0, \sigma_2 \geq 0, \ldots, \sigma_n \geq 0 \Rightarrow \sigma_0 \leq 0$$

$$S2: \sigma_0 + \sum_{i=1}^{N} \tau_i \sigma_i \leq 0$$

with $\tau_i \geq 0$, $i = 1, \ldots, N$. It is obvious that $S2 \Rightarrow S1$. But it is not always true that $S1 \Rightarrow S2$. It is called lossless if $S1 \Leftrightarrow S2$.

We are concerned about the case when σ_k take the quadratic forms as

$$\sigma_k(f) = \langle \Phi_k f, f \rangle, \quad k = 0, 1, \ldots, N. \tag{2}$$

with $\Phi_k^* = \Phi_k$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using the S-procedure to establish stability

Suppose we have chosen a Lyapunov function candidate $V > 0$.

If we denote $\sigma_0 := \dot{V}(x) < 0$ and use $\sigma_i \geq 0$ with $i = 1, \ldots, N$ to describe the N quadratic constraints derived from the nonlinearity in the Lure system. Then a sufficient condition for the system to be stable is the satisfaction of the inequality:

$$\sigma_0 + \sum_{i=1}^{N} \tau_i \sigma_i < 0 \qquad (3)$$

with $\tau_i \geq 0$ and $i = 1, \ldots, N$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using the S-procedure to establish stability

We are interested in the question: how to reduce the conservatism of the stability criterion.

We have two options in this direction:

- Choose different Lyapunov function candidates;
- Use more constraints describing the nonlinearity.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Circle Criterion

Suppose $\phi \in [0, K]$. Then the sector condition is

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \Pi_c \begin{bmatrix} x \\ u \end{bmatrix} \geq 0$$

with

$$\Pi_c = \begin{bmatrix} 0 & -C^T K^T \\ -KC & -2I \end{bmatrix}$$

Choose a Lyapunov function $V(x) = x^T P x$, with $P > 0$. The system is g.a.s., if under the sector condition, we have

$$\dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T \Pi_v \begin{bmatrix} x \\ u \end{bmatrix} < 0$$

with

$$\Pi_v = \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Circle Criterion

Then the Circle criterion can be derived by the S-procedure:

$$\Pi_v + \Pi_c = \begin{bmatrix} A^T P + PA & PB - C^T K^T \\ B^T P - KC & -2I \end{bmatrix} < 0 \qquad (4)$$

This is a linear matrix inequality (LMI).
The feasibility of an LMI can be tested easily using some exiting LMI solvers, such as LMILAB (a toolbox of MATLAB), SEDUMI, etc.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Popov Criterion – with positive multiplier

Choose the candidate Lyapunov function (Lure and Postnikov type) as

$$V(x) = x^T P x + 2r \int_0^y \phi(\tau) d\tau \tag{5}$$

with $r > 0$ and $P > 0$. The derivative of $V(x)$ is

$$\dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T (\Pi_v + r\Pi_p) \begin{bmatrix} x \\ u \end{bmatrix} < 0 \tag{6}$$

with

$$\Pi_p = - \begin{bmatrix} 0 & A^T C^T \\ CA & CB + B^T C^T \end{bmatrix} \tag{7}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Popov Criterion – with positive multiplier

Then the Popov criterion can be derived by the S-procedure:

$$\Pi_v + \Pi_c + r\Pi_p = \begin{bmatrix} A^T P + PA & PB - C^T K^T - rA^T C^T \\ B^T P - KC - rCA & -2I - r(CB + B^T C^T) \end{bmatrix} < 0 \tag{8}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Popov Criterion – with indefinite multiplier (SISO)

Choose the candidate Lyapunov function as

$$V(x) = x^T P x + 2r_+ \int_0^y \phi(\tau) d\tau + 2r_- \int_0^y [K\tau - \phi(\tau)] d\tau \quad (9)$$

with $r_+ > 0$, $r_- > 0$ and $P > 0$. The derivative of $V(x)$ is

$$\dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T (\Pi_v + r_+ \Pi_{p+} + r_- \Pi_{p-}) \begin{bmatrix} x \\ u \end{bmatrix} < 0 \quad (10)$$

with

$$\Pi_{p+} = - \begin{bmatrix} 0 & A^T C^T \\ CA & CB + B^T C^T \end{bmatrix} \quad (11)$$

$$\Pi_{p-} = - \begin{bmatrix} (KC)^T CA + (CA)^T KC & (KC)^T CB \\ (CB)^T KC & 0 \end{bmatrix} \quad (12)$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Popov Criterion – with indefinite multiplier (SISO)

Then the Popov criterion can be derived by the S-procedure:

$$\Pi_v + \Pi_c + r_+ \Pi_{p+} + r_- \Pi_{p-} < 0 \quad \text{with } r_+, r_- \geq 0 \qquad (13)$$

which can be (in some cases) shown equivalent to

$$\Pi_v + \Pi_c + r \Pi_{p+} < 0 \qquad (14)$$

with $r \in \mathbb{R}$.

- For SISO case, the equivalence can be shown by loop transformation [Yakubovich(1967)];
- Diagonal MIMO case is shown in [Yakubovich(1967), Park(1997)];
- Coupled MIMO case is shown in [Heath and Li(2009)].

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Choice of Lyapunov Function candidates

To summarize, we have used the same sector bound condition and chosen three different Lyapunov functions for three cases:

- Circle

$$V(x) = x^T P x$$

- Popov (positive multiplier)

$$V(x) = x^T P x + 2r \int_0^y \phi(\tau) d\tau$$

- Popov (indefinite multiplier)

$$V(x) = x^T P x + 2r_+ \int_0^y \phi(\tau) d\tau + 2r_- \int_0^y [K\tau - \phi(\tau)]^T d\tau$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

The Criteria in Frequency Domain

Using the **K**alman- **Y**akubovich-**P**opov (KYP) Lemma, these criteria in the LMI forms can be transformed into frequency domain:

- Circle

$$\text{Re}\{KG(j\omega) + I\} \geq 0 \quad \forall \omega \in \mathbb{R}$$

- Popov (positive multiplier)

$$\text{Re}\{(K + rj\omega I)G(j\omega) + I\} \geq 0 \quad \forall \omega \in \mathbb{R}$$

with $r \geq 0$.

- Popov (indefinite multiplier)

$$\text{Re}\{(K + rj\omega I)G(j\omega) + I\} \geq 0 \quad \forall \omega \in \mathbb{R}$$

with $r \in \mathbb{R}$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
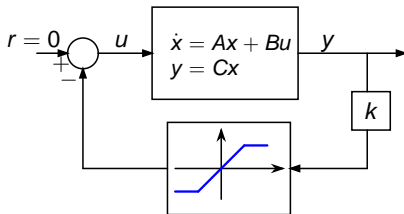Conclusion
Bibliography

Example 1

Figure: Absolute stability example

$$A = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
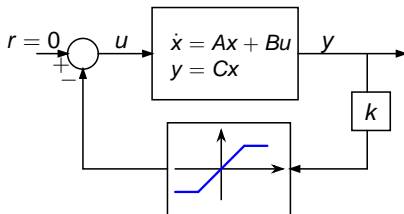Bibliography

Example 1

Figure: Absolute stability example

The rang of $k > 0$ that guarantees the stability of the system:

- Circle: $k \leq 8.12$;
- Popov – positive multiplier: $k \leq 8.12$;
- Popov – indefinite multiplier: $k \leq 8.9$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using the S-procedure to establish stability

We are interested in the question: how to reduce the conservatism of the stability criterion.

We have two options in this direction:

- Choose different Lyapunov function candidates;
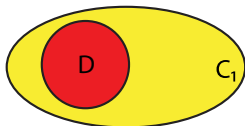- Use more constraints describing the nonlinearity.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

The constraints of the nonlinearity

- D – The nonlinearity.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

The constraints of the nonlinearity

- D – The nonlinearity.
- C_1 – The constraint $\sigma_1 \geq 0$.

Introduction of Lure systems
Stability establishment
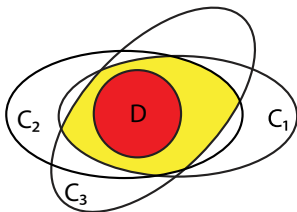Explore more properties using quadratic programme
Conclusion
Bibliography

The constraints of the nonlinearity

- D – The nonlinearity.
- C_1, C_2 – The constraints $\sigma_i \geq 0$ with $i = 1, 2$ respectively.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
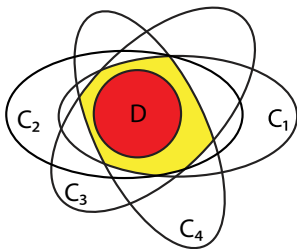Conclusion
Bibliography
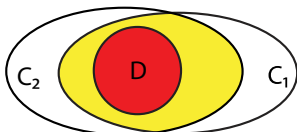
The constraints of the nonlinearity

- D – The nonlinearity.
- C_1, C_2, C_3 – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3$ respectively.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
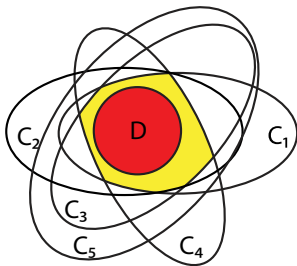Conclusion
Bibliography

The constraints of the nonlinearity

- D – The nonlinearity.
- C_1, C_2, C_3, C_4 – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3, 4$ respectively.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

The constraints of the nonlinearity

- D – The nonlinearity.
- C_1, C_2, C_3, C_4, C_5 – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3, 4, 5$ respectively.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Outline

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities
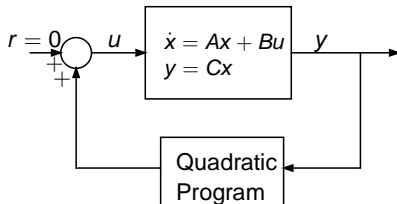
The nonlinearity $\phi(\cdot)$ can be expressed by a quadratic program (QP) [Primbs(2001)]:

$$u(t) = \phi(y(t)) = \arg\min_{\tilde{u}} \frac{1}{2}\tilde{u}^T H \tilde{u} + \tilde{u}^T y(t) \tag{15}$$
$$\text{subject to } L\tilde{u} + My \preceq b$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Common nonlinearities

Saturation

Figure: Saturation

Saturation nonlinearity can be described by

$$u = \arg\min_{\tilde{u}} \frac{1}{2}(\tilde{u} - y)^2 \tag{16}$$
$$\text{s.t.} |\tilde{u}| \leq 1$$

which corresponds to the QP with $H = 1$, $F = -1$, $L = 1$, $N = 0$ and $b = 1$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Common nonlinearities

Deadzone
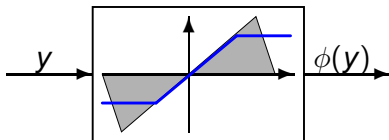

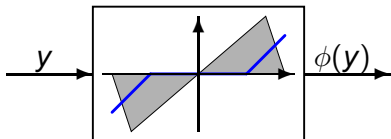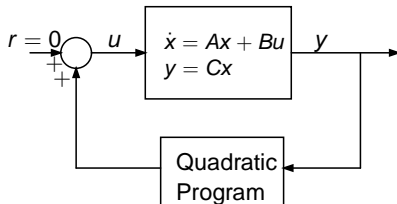Figure: Saturation

Saturation nonlinearity can be described by

$$u = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^2 \tag{17}$$
$$\text{s.t.} |\tilde{u} - y| \le 1$$

which corresponds to the QP with $H = 1$, $F = 0$, $L = 1$, $N = -1$ and $b = 1$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

The nonlinearity $\phi(\cdot)$ can be expressed by a quadratic program (QP) [Primbs(2001)]:

$$u(t) = \phi(y(t)) = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T y(t) \tag{18}$$
$$\text{subject to } L\tilde{u} + My \preceq b$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

We can derive its KKT (Karush-Kuhn-Tucker) conditions:

$$Hu + Fy + L^T\lambda = 0 \tag{19a}$$
$$Lu + My + s = b \tag{19b}$$
$$\lambda^T s = 0 \tag{19c}$$
$$\lambda \succeq 0 \tag{19d}$$
$$s \succeq 0 \tag{19e}$$

where λ is the Lagrangian multiplier and s is the slack variable. The KKT conditions are the necessary and sufficient conditions for the solution of the QP to be optimal.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

From these KKT conditions and there first derivatives, we can
derive three quadratic conditions
[Li et al.(2007)Li, Heath, and Lennox]
[Li et al.(2008)Li, Heath, and Lennox]:

$$(u + L^\dagger My)^T(Hu + Fy) - ((I - LL^\dagger)My)^T\lambda \leq 0 \qquad (20a)$$

$$(\dot{u} + L^\dagger M\dot{y})^T(Hu + Fy) - ((I - LL^\dagger)M\dot{y})^T\lambda = 0 \qquad (20b)$$

$$(\dot{u} + L^\dagger M\dot{y})^T(H\dot{u} + F\dot{y}) - ((I - LL^\dagger)M\dot{y})^T\lambda = 0 \qquad (20c)$$

where $L^\dagger := (L^T L)^{-1}L^T$. The columns of L are linearly
independent.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

From these KKT conditions and there first derivatives, we can derive three quadratic conditions
[Li et al.(2007)Li, Heath, and Lennox]
[Li et al.(2008)Li, Heath, and Lennox]:

$$(u + L^{\dagger}My)^T(Hu + Fy) - ((I - LL^{\dagger})My)^T\lambda \leq 0 \qquad (20a)$$

$$(\dot{u} + L^{\dagger}M\dot{y})^T(Hu + Fy) - ((I - LL^{\dagger})M\dot{y})^T\lambda = 0 \qquad (20b)$$

$$(\dot{u} + L^{\dagger}M\dot{y})^T(H\dot{u} + F\dot{y}) - ((I - LL^{\dagger})M\dot{y})^T\lambda = 0 \qquad (20c)$$

where $L^{\dagger} := (L^T L)^{-1}L^T$. The columns of L are linearly independent.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

Assuming $M = LN$, three quadratic conditions can be further simplified:

$$(u + Ny)^T(Hu + Fy) \leq 0 \tag{21a}$$

$$(\dot{u} + N\dot{y})^T(Hu + Fy) = 0 \tag{21b}$$

$$(\dot{u} + N\dot{y})^T(H\dot{u} + F\dot{y}) = 0 \tag{21c}$$

where $y = Cx$ and $\dot{y} = C\dot{x} = CAx + CBu$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Common nonlinearities

Saturation

Figure: Saturation

Its quadratic constraints are:

$$u(u - y) \leq 0 \tag{22}$$

$$\dot{u}(u - y) = 0 \tag{23}$$

$$\dot{u}(\dot{u} - \dot{y}) = 0 \tag{24}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
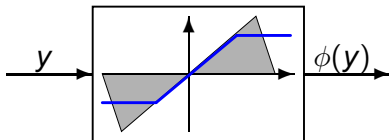Bibliography

Common nonlinearities

Deadzone
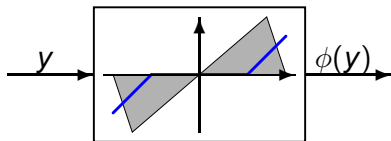
Figure: Saturation

Its quadratic constraints are:

$$u(u - y) \leq 0 \qquad (25)$$
$$u(\dot{u} - \dot{y}) = 0 \qquad (26)$$
$$\dot{u}(\dot{u} - \dot{y}) = 0 \qquad (27)$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

Assuming $M = LN$, three quadratic conditions can be further simplified:

$$(u + Ny)^T(Hu + Fy) \leq 0 \qquad (28a)$$

$$(\dot{u} + N\dot{y})^T(Hu + Fy) = 0 \qquad (28b)$$

$$(\dot{u} + N\dot{y})^T(H\dot{u} + F\dot{y}) = 0 \qquad (28c)$$

where $y = Cx$ and $\dot{y} = C\dot{x} = CAx + CBu$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

Denote $v := [x^T, u^T, \dot{u}^T]^T$, then the three quadratic conditions can be expressed as

$$v^T \Pi_1 v \geq 0 \tag{29a}$$
$$v^T \Pi_2 v = 0 \tag{29b}$$
$$v^T \Pi_3 v = 0 \tag{29c}$$

with

$$\Pi_1 = -\operatorname{He}\left(\begin{bmatrix} NC & I & 0 \end{bmatrix}\begin{bmatrix} FC & H & 0 \end{bmatrix}\right) \tag{30a}$$
$$\Pi_2 = \operatorname{He}\left(\begin{bmatrix} NCA & NCB & I \end{bmatrix}\begin{bmatrix} FC & H & 0 \end{bmatrix}\right) \tag{30b}$$
$$\Pi_3 = \operatorname{He}\left(\begin{bmatrix} NCA & NCB & I \end{bmatrix}\begin{bmatrix} FCA & FCB & H \end{bmatrix}\right) \tag{30c}$$

Here $\operatorname{He}(M) = M^* + M$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

Choose a candidate Lyapunov function

$$V(x, u) = [x^T, u^T]P[x^T, u^T]^T \tag{31}$$

with $P > 0$. The first derivative of V is

$$\dot{V}(x, u) = \begin{bmatrix} x \\ u \\ \dot{u} \end{bmatrix}^T \underbrace{\begin{bmatrix} A^T P_{11} + P_{11}A & A^T P_{12} + P_{11}B & P_{12} \\ P_{12}^T A + B^T P_{11} & P_{12}^T B + B^T P_{12} & P_{22} \\ P_{12}^T & P_{22} & 0 \end{bmatrix}}_{\Pi_v} \begin{bmatrix} x \\ u \\ \dot{u} \end{bmatrix} \tag{32}$$

that is

$$\dot{V}(x, u) = v^T \Pi_v v \tag{33}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Using quadratic programme to express nonlinearities

The system is stable if there exists a positive definite matrix
$P > 0$ such that the following LMI is satisfied:

$$\Pi_v + \sum_{i=1}^{3} r_i \Pi_i < 0 \tag{34}$$

with scalar variables $r_1 \geq 0$, $r_2, r_3 \in \mathbb{R}$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography
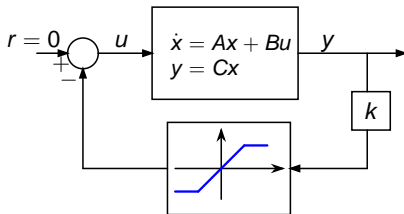
Example 1 – continued

Figure: Absolute stability example

$$A = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
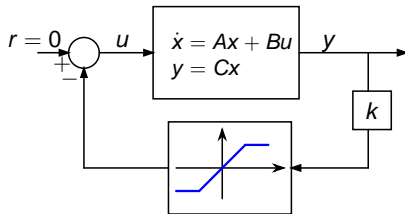Conclusion
Bibliography

Example 1 – continued

Figure: Absolute stability example

The rang of $k > 0$ that guarantees the stability of the system:

- Circle: $k \leq 8.12$;
- Popov (indefinite multiplier): $k \leq 8.9$;
- QP method: $k < +\infty$.
- Zames-Falb multiplier: $k < +\infty$;

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Example 2

- Two cases for the nonlinearity ϕ: saturation or deadzone;
- Two cases for the plant

$$G_1(s) = \frac{1}{s^4 + s^3 + 8s^2 + 2s + 1}$$

$$G_2(s) = \frac{1}{s^4 + s^3 + 8s^2 + 3s + 1}$$

- Two Lyapunov function candidates:

$$V_1(x, u) = [x^T, u^T]P[x^T, u^T]$$

$$V_2(x, u) = [x^T, u^T]P[x^T, u^T] + r\int_0^y \phi(\tau)d\tau \quad \text{with } r \geq 0$$

Note: V_2 is inspired by the Lyapunov function corresponding to the Popov criterion.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Example 2

Table: A comparison of the existing approaches and the new approach: continuous case. In each case the maximum value of K for which the system is guaranteed stable is shown.

ϕ	Plant	Circle	Popov	QP (V_1)	QP (V_2)
sat	G_1	1.69	4.03	3.98	9.63
dz	G_2	2.94	5.41	6.93	11.56
dz	G_1	1.69	1.69	10.16	10.16

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Extension to stability analysis of model predictive control (MPC)

Discrete version

The stability analysis based on quadratic programme has a discrete time version.

Extension to MPC analysis

MPC resolves a constrained optimization problem at each sampling instant. The optimization can be expressed as a QP. Hence we can extend this method to MPC analysis.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Extension to stability analysis of model predictive control (MPC)

MPC controller – minimization of cost function

An optimization problem is resolved at each sampling instant k

$$U_k^* = \arg \min_{U_k} x_{k+N_p}^T S x_{k+N_p} + \sum_{i=0}^{N_p-1} \left[x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right]$$

$$\text{s.t.} L_k u_k \preceq b_k \text{ with } b_k \succeq 0 \tag{35}$$

where $U_k = [u_k^T, u_{k+1}^T, \ldots, u_{k+N_p-1}^T]$ is the optimizer. Only the first element of U_k^*, i.e. u_k^*, is used as the control input.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Extension to robustness analysis of model predictive control (MPC)

MPC controller – quadratic programme

If only input constraints are considered, the MPC optimization problem can be written as a quadratic programme

$$U_k^* = \arg\min_{U_k} U_k^T H U_k + F_k^T U_k$$

$$\text{s.t.} L U_k \preceq b \text{ with } b \succeq 0$$

(36)

where H is a positive definite Hessian matrix, and F_k contains the estimated or measured state x_k.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

MPC framework

Figure: MPC framework

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography
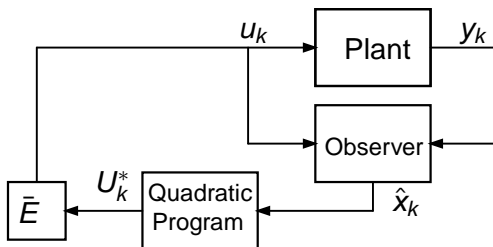
Example 3 – Robustness analysis of MPC

Figure: MPC framework

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Example 3 – Robustness analysis of MPC

An MPC example with uncertainties
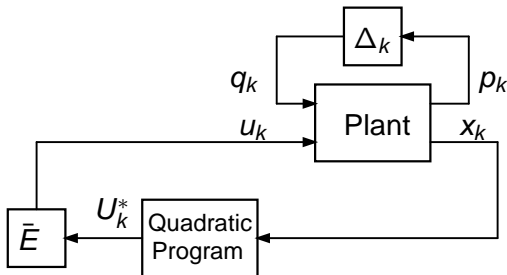
Suppose a plant with unstructured uncertainties is

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k$$
$$p_k = Cx_k + D_u u_k + D_w w_k$$
$$w_k = \Delta_k p_k$$

where Δ_k satisfies

$$\|\Delta_k\|_2 \leq 1$$

with

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \qquad B_u = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad B_w = \begin{bmatrix} \theta & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D_u = 0 \qquad D_w = 0$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Example 3 – Robustness analysis of MPC

The plant is subject to input constraints $|u| \leq 1$.
The cost function is chosen as

$$U_k^* = \arg \min_{U_k} x_{k+N_p} S x_{k+N_p} + \sum_{i=0}^{N_p-1} \left[x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right]$$

$$\text{s.t.} L_k u_k \preceq b_k \text{ with } b_k \succeq 0$$

(37)

with

$$Q = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 3/2 \end{bmatrix} \qquad R = 1 \qquad S = Q \qquad N_p = 3$$

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Example 3 – Robustness analysis of MPC

The Lyapunov function is chosen to be

$$V = [x_k^T, U_k^T]^T P[x_k^T, U_k^T]$$

We use the three constraints derived from the QP and the quadratic constraints from the uncertainty $\|\Delta_k\| \leq 1$ to establish stability by the S-procedure.

The range of θ for the system to be stable is $0 \leq \theta \leq 0.19$.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Outline

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.

- Two directions for reducing the stability conservatism:
  - Choose suitable Lyapunov function candidate;
  - Exploit more useful constraints for the nonlinearity.

- Quadratic program can be used for the stability analysis of Lure system.

- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
  - Choose suitable Lyapunov function candidate;
  - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
    - Choose suitable Lyapunov function candidate;
    - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
  - Choose suitable Lyapunov function candidate;
  - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
    - Choose suitable Lyapunov function candidate;
    - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
    - Choose suitable Lyapunov function candidate;
    - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
- QP method can be further extended for the robust stability analysis of MPC systems.

Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

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Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
Bibliography

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Introduction of Lure systems
Stability establishment
Explore more properties using quadratic programme
Conclusion
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